Electromagnetic Waves and Oscillations at the Surface of Conductors.

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GENERAL EQUATIONS.

In the following paper I have worked out a few cases of electromagnetic waves and the oscillations of electricity on a conducting body, such as may be useful in the further understanding of alternating currents and the subject of electricity generally.

Of course all calculations must be based on Maxwell's equations. In these equations occur two quantities, J and ψ, which caused remarks by Sir Wm. Thomson and others at the British Association meeting in Bath. Maxwell has already given the reasons for rejecting ψ, and has shown that neither J nor ψ enter into the theory of waves. In order, however, that there shall be no propagation of free electricity in a non-conductor, the components of the electric force must satisfy the equation of continuity, and this leads to components of the vector potential satisfying the same equation, and $J=0$ therefore. I have satisfied myself that there is absolutely no loss of generality from these changes. Hence I write the equations as follows: F, G, H being the components of the vector potential, u, v, w of the electric current, P, Q, R of the electric force, $\bar{a}$, $\bar{b}$, $\bar{c}$ of the magnetic induction, $\varkappa$ the specific inductive capacity, μ the magnetic permeability, C the conductivity, t the time and σ the surface density.

$$\bar{a}=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}; \qquad P=-\frac{dF}{dt}$$
$$\bar{b}=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}; \qquad Q=-\frac{dG}{dt}$$
$$\bar{c}=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}; \qquad R=-\frac{dH}{dt}\cdot$$

$$u = -\frac{1}{4\pi\mu}\Delta^2 F = -\left(C + \frac{\varkappa}{4\pi}\frac{d}{dt}\right)\frac{dF}{dt}$$
$$v = -\frac{1}{4\pi\mu}\Delta^2 G = -\left(C + \frac{\varkappa}{4\pi}\frac{d}{dt}\right)\frac{dG}{dt}$$
$$w = -\frac{1}{4\pi\mu}\Delta^2 H = -\left(C + \frac{\varkappa}{4\pi}\frac{d}{dt}\right)\frac{dH}{dt}$$
$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0.$$

Inside a good conductor, such as a metal, Maxwell has shown that $\varkappa$ can be neglected in comparison with the conductivity, and the equations take the form of the diffusion equations for heat.

The conditions at a surface of separation of two media are as follows, though all the equations are not independent, l, m and n being the direction cosines of the normal to the surface:

$$\begin{aligned}
&(\bar{a} - \bar{a}_1)\, l + (\bar{b} - \bar{b}_1)\, m + (\bar{c} - \bar{c}_1)\, n = 0, \\
&(u - u_1)\, l + (v - v_1)\, m + (w - w_1)\, n = 0, \\
&(\varkappa P - \varkappa_1 P_1)\, l + (\varkappa Q - \varkappa_1 Q_1)\, m + (\varkappa R - \varkappa_1 R_1)\, n = 4\pi\sigma,
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\bar{b}}{\mu} - \frac{\bar{b}_1}{\mu_1}\right) n - \left(\frac{\bar{c}}{\mu} - \frac{\bar{c}_1}{\mu_1}\right) m &= 0, \\
\left(\frac{\bar{c}}{\mu} - \frac{\bar{c}_1}{\mu_1}\right) l - \left(\frac{\bar{a}}{\mu} - \frac{\bar{a}_1}{\mu_1}\right) n &= 0, \\
\left(\frac{\bar{a}}{\mu} - \frac{\bar{a}_1}{\mu_1}\right) m - \left(\frac{\bar{b}}{\mu} - \frac{\bar{b}_1}{\mu_1}\right) l &= 0,
\end{aligned}$$

$$\begin{aligned}
(Q - Q_1)\, n - (R - R_1)\, m &= 0, \\
(R - R_1)\, l - (P - P_1)\, n &= 0, \\
(P - P_1)\, m - (Q - Q_1)\, l &= 0.
\end{aligned}$$

The electrostatic energy of any volume is

$$\frac{\varkappa}{8\pi}\iiint (P^2 + Q^2 + R^2)\, dxdydz,$$

and the electromagnetic energy is

$$\frac{1}{8\pi\mu}\iiint (\bar{a}^2 + \bar{b}^2 + \bar{c}^2)\, dxdydz.$$

The equations of Maxwell, applied to periodic disturbance, indicate that all the vector quantities have only two directions in space at any point. The vector potential, the electric current, the electric force and the electric induction being in one direction, and the magnetic force and magnetic induction being in another direction, the cosine of the angle between these directions being

$$\frac{F\bar{a}+G\bar{b}+H\bar{c}}{\sqrt{(\bar{a}^2+\bar{b}^2+\bar{c}^2)(F^2+G^2+H^2)}}.$$

A system of surfaces can always be drawn containing these two vectors, as in hydrodynamics in the case of steady flow for the vortex lines and lines of flow, and these surfaces constitute the wave surface when the motion is periodic. In any case we may call them by this name. In the case of periodic motion, the flow of energy is perpendicular to this surface for plane waves at least, and Prof. Poynting has carried out the idea for all cases and supposes the energy always to flow perpendicular to this surface.

At the surface of a perfect conductor the conditions are much simplified. The disturbance penetrates only an infinitely small distance into the surface, and consists of a current sheet whose components per unit length along arc of cross section can be designated by U, V, W. The surface conditions are then transformed into the following:

$$U=\frac{1}{4\pi\mu}\{m\bar{c}-n\bar{b}\},$$

$$V=\frac{1}{4\pi\mu}\{n\bar{a}-l\bar{c}\},$$

$$W=\frac{1}{4\pi\mu}\{l\bar{b}-m\bar{a}\},$$

$$\therefore\begin{cases}\bar{a}U+\bar{b}V+\bar{c}W=0,\\ lU+mV+nW=0,\end{cases}$$

$$lu+mv+nw=-\left\{\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right\}=-\frac{d\sigma}{dt},$$

$$l\bar{a}+m\bar{b}+n\bar{c}=0,$$

$$lP+mQ+nR=\frac{4\pi\sigma}{\varkappa},$$

$$mR-nQ=0,$$
$$nP-lR=0,$$
$$lQ-mP=0.$$

From these equations we see that the electrical vectors, such as the vector potential, electric force and electric current, in the medium must be perpendicular to the perfectly conducting surface, and that the magnetic force and induction must be parallel to the surface. The wave surface must then be perpendicular to the conducting surface, and the currents in the surface are in the direction of the normal to the wave surface. Thus the system is orthogonal at the surface.

Selecting a complete orthogonal system, one surface being the wave surface, we can replace the surface containing the magnetic induction $\bar{a}$, $\bar{b}$, $\bar{c}$ by a perfectly conducting surface.

The solution of cases of progressive waves, as well as stationary waves and electric oscillations on conductors, are all dependent on these general equations. One general fact with regard to electric oscillations is to be noted, and that is, that, as in the case of sound, the period of oscillation of an electric system is proportional to the linear dimensions of the system.

Two Dimensions.

Suppose the waves to move in the direction of the axis of X without any change of form, but with or without damping, which would influence the amplitude of the disturbance. Then the disturbance can be expressed by the equations

$$
\begin{aligned}
F &= -\frac{1}{a} L\varepsilon^{ax+ct}, & \bar{a} &= T\varepsilon^{ax+ct}, \\
G &= M\varepsilon^{ax+ct}, & \bar{b} &= \frac{1}{a}\left(\frac{\partial T}{\partial y} - AN\right)\varepsilon^{ax+ct}, \\
H &= N\varepsilon^{ax+ct}, & \bar{c} &= \frac{1}{a}\left(\frac{\partial T}{\partial z} + AM\right)\varepsilon^{ax+ct},
\end{aligned}
$$

where L, M, N are functions of y and z and

$$A = c\mu\,(4\pi C + cx); \quad L = \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}; \quad T = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z};$$
$$\Delta^2 M + (a^2 - A)\, M = 0; \quad \Delta^2 N + (a^2 - A)\, N = 0.$$

In these equations in general $a = \alpha + i\beta$ and $c = h - ib$, where α is the distance damping factor, h is the time damping factor, the velocity of the wave is $\frac{b}{\beta}$ and $b = \pi\nu$, where ν is the number of alternations per second.

A special case is given by $F = 0$ and $a^2 - A = 0$, which is applicable to waves on perfectly conducting cylinders.

These conditions give

$$\frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} = 0,$$

$$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0.$$

Hence M and N are conjugate functions of y and z. If ϕ and ψ are conjugate functions of y and z, we can therefore write

$$M = \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial z},$$

$$N = \frac{\partial \phi}{\partial z} = -\frac{\partial \psi}{\partial y},$$

$$\therefore\ F = 0, \qquad \bar{a} = 0,$$

$$G = \frac{\partial \phi}{\partial y}\, \varepsilon^{ax+ct}, \qquad \bar{b} = a\frac{\partial \psi}{\partial y}\, \varepsilon^{ax+ct},$$

$$H = \frac{\partial \phi}{\partial z}\, \varepsilon^{ax+ct}, \qquad \bar{c} = a\frac{\partial \psi}{\partial z}\, \varepsilon^{ax+ct}.$$

Hence the electric quantities lie on the curve $\psi =$ const. and the magnetic ones on $\phi =$ const. Therefore the conditions at the surface of a perfect conductor are satisfied for the cylindrical surfaces $\phi =$ const.

In general, we take two cylinders, ϕ' and ϕ'', so as to account for the direct and return currents and for the $+$ and $-$ charges.

The surface density on $\phi =$ const. is found to be

$$\sigma = \frac{-c\kappa}{4\pi}\, h\varepsilon^{ax+ct},$$

where

$$h^2 = \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2$$

and the electric current

$$U_1 = a\frac{h}{4\pi\mu}\, \varepsilon^{ax+ct}.$$

Hence we have the remarkable theorems:

Theorem I. *The electric density on the cylinders is proportional to the density which they would have if they were charged with electricity at rest; and the electric currents are distributed in the same proportion, although not the same phase.*

This leads to

Theorem II. *In the case of perfect conductors we have* $LC = \frac{1}{v^2}$, *where* L *is the self-induction and* C *the capacity per unit of length, and* v *the ratio of the units or the velocity of light.*

This is best proved by considering the energy of the system per unit of length as derived by the formulae above and by the ordinary ones in terms of the capacity and self-induction. Thus we have

$$\frac{1}{2}\frac{\left(\int \sigma ds\right)^2}{C} = \frac{\varkappa c^2}{8\pi}\int\int\int h^2 (\varepsilon^{ax+ct})^2\, dxdydz,$$

$$\frac{1}{2}L\left(\int U_1 ds\right)^2 = \frac{a^2}{8\pi\mu}\int\int\int h^2 (\varepsilon^{ax+ct})^2\, dxdydz,$$

where ds is an element of the curve ϕ constant, and the line integrals are taken around the curve ϕ' or ϕ'', and the volume integrals between ϕ' and ϕ''. Whence, substituting the values of σ and U_1 and dividing one equation by the other, the theorem is proved.

In the case of imperfect conductors and long waves, the currents sink below the surface and the self-induction becomes greater than $\frac{1}{Cv^2}$.

It is to be noted that rapid alternation of the current to some extent takes the place of improved conductivity in causing a superficial distribution of the current. Indeed the equations only contain the conductivity multiplied by the number of alternations per second, although the number of alternations per second enters separately. Within certain limits, therefore, the following is correct:

Starting with very good conductors and very long waves, the electric current will be uniformly distributed throughout the section of the conductors. As the waves become shorter and the number of vibrations per second greater, the distribution of currents changes and finally they are entirely on the surface of the conductors and distributed very nearly like the density of electricity, pro-

vided the conductors were charged with electricity at rest. The waves must, however, be long as compared with the sectional area of the cylinders.*

Plane Waves at the Surface of Separation of Two Media.

At the surface of a liquid or of an elastic solid we may have waves propagated according to known laws. So, at the surface separating two media, there may be waves whose theory will now be given. In this investigation a, b, c, e and A, B will be complex constants. Let

$$F = Ae\varepsilon^{ax+ey+ct},$$
$$G = -Aa\varepsilon^{ax+ey+ct},$$
$$H = B\,\varepsilon^{ax+ey+ct},$$

which satisfy the equation of continuity. Maxwell's equation gives also

$$-\mu c(4\pi C - \varkappa c) + a^2 + e^2 = 0.$$

Let the plane $y = 0$ separate the two media of conductivities C and C_1 and specific inductive capacities and magnetic permeabilities $\varkappa\mu$ and $\varkappa_1\mu_1$. Maxwell's equation and the surface conditions then give

$$a = a_1, \qquad c = c_1,$$
$$a^2 + e^2 = \mu c(4\pi C + \varkappa c), \quad a^2 + e_1^2 = \mu_1 c(4\pi C_1 + \varkappa_1 c),$$
$$\frac{A}{\mu}(a^2 + e^2) = \frac{A_1}{\mu_1}(e_1^2 + a^2), \qquad B = B_1,$$
$$Ae = A_1 e_1, \qquad \frac{e}{\mu} = \frac{e_1}{\mu_1}.$$

There are two principal cases—

1*st.—Magnetic Waves.*

The condition is $A = A_1 = 0$,

$$F = G = 0, \qquad F_1 = G_1 = 0.$$
$$H = B\varepsilon^{ax+ey+ct}, \qquad H_1 = B\varepsilon^{ax+\frac{\mu_1}{\mu}ey+ct}.$$

When there is no time damping at the source of the wave, we can pass to a real solution as follows:

$$a = \alpha + i\beta, \quad e = \gamma + i\delta, \quad c = -ib = -i\pi\nu.$$

*The distribution of currents is superficial for moderate conductivity and short waves. Hence, inside the conductors, the electric and magnetic forces must be zero, and this leads to the same superficial distribution of electric currents and surface density which, when the section of the cylinder is small compared with the wave length, becomes that of electricity at rest.

The conditions from Maxwell's equations give us the values of these in terms of b.

$$\alpha^2 = b\,\frac{\mu\mu_1}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1^{\mu}-C_{\mu_1})^2+b^2(\varkappa_1^{\mu}-\varkappa_{\mu_1})^2}+b(\varkappa_1^{\mu}-\varkappa_{\mu_1})\right\},$$

$$\beta^2 = b\,\frac{\mu\mu_1}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1^{\mu}-C_{\mu_1})^2+b^2(\varkappa_1^{\mu}-\varkappa_{\mu_1})^2}-b(\varkappa_1^{\mu}-\varkappa_{\mu_1})\right\},$$

$$\gamma^2 = b\,\frac{\mu^2}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1\mu_1-C\mu)^2+b^2(\varkappa_1\mu_1-\varkappa\mu)^2}+b(\varkappa_1\mu_1+\varkappa\mu)\right\},$$

$$\delta^2 = b\,\frac{\mu^2}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1\mu_1-C\mu)^2+b^2(\varkappa_1\mu_1-\varkappa\mu)^2}-b(\varkappa_1\mu_1-\varkappa\mu)\right\},$$

where α and β must have opposite signs. For the second medium

$$\alpha_1=\alpha;\quad \beta_1=\beta;\quad \gamma_1=\frac{\mu_1}{\mu}\gamma;\quad \delta_1=\frac{\mu_1}{\mu}\delta.$$

When $\mu_1=\mu$, these all become infinite and no wave is propagated. Hence they depend on the difference of magnetic properties. The magnetic disturbance is in the plane xy and the electrical disturbance in the direction of the axis of z. The wave advances in direction of x with a velocity $\frac{b}{\beta}$ and a damping factor α. The waves in both media die out quickly as one passes away from the surface according to the damping factors γ and γ_1. So that the wave is confined to the surface in the same sense that a water wave is confined to the surface of water.

The case of greatest consequence is that of iron and air for which

$$C=0,\quad \mu=1,\quad \varkappa=\frac{1}{9\times10^{20}};\quad C_1=\frac{1}{10000},\quad \mu_1=1000,\quad \varkappa_1=0,$$

all on the c. g. s. system. These give very nearly

$$-\alpha=\beta=-\gamma=\delta=\pi\sqrt{\frac{2C_1\nu}{\mu}}\ \text{ and }\ \gamma_1=-\delta_1=\pi\sqrt{2C_1\mu_1\nu}.$$

If λ is the wave length of a complete wave, ν the number of reversals per second and V the velocity of the waves, we have

$$b=\frac{2\pi V}{\lambda}=\beta V=\pi V=\beta\frac{\nu\lambda}{2}.$$

Hence the disturbance is given by the following quantities:

Vector Potential.

In Air.

$$F=G=0,$$
$$H=B\varepsilon^{-\beta(x+y)}\cos\{\beta(x+y)-\pi\nu t\}.$$

In the Magnetic Metal.

$$F_1 = G_1 = 0,$$
$$H_1 = B\varepsilon^{-\beta(x+\mu_1 y)} \cos \{\beta (x + \mu_1 y - \pi\nu t\}.$$

Magnetic Induction.

In Air.

$$\bar{a} = - B\varepsilon^{-\beta(x+y)}\beta \{\sin [\beta (x + y) - \pi\nu t] + \cos [\beta (x + y) - \pi\nu t]\},$$
$$\bar{b} = + B\varepsilon^{-\beta(x+y)}\beta \{\sin [\beta (x + y) - \pi\nu t] + \cos [\beta (x + y) - \pi\nu t]\},$$
$$\bar{c} = 0.$$

In the Magnetic Metal.

$$\bar{a}_1 = - B\varepsilon^{-\beta(x+\mu_1 y)}\beta\mu_1 \{\sin [\beta (x + \mu_1 y) - \pi\nu t] + \cos [\beta (x + \mu_1 y) - \pi\nu t]\},$$
$$\bar{b}_1 = + B\varepsilon^{-\beta(x+\mu_1 y)}\beta \quad \{\sin [\beta (x + \mu_1 y) - \pi\nu t] + \cos [\beta (x + \mu_1 y) - \pi\nu t]\},$$
$$\bar{c}_1 = 0.$$

The waves which proceed outward from the surface into the two media are therefore of the same type as the heat waves sent downward into the earth from the periodic heating of the sun, and may be called diffusion waves. The component along the surface is of the same type. In this type the real and imaginary parts of the coefficients of x or y are equal, and therefore the amplitude of the wave is decreased to $\frac{1}{535}$th of its value in going a complete wave length or $\frac{1}{23}$rd in going half a wave length. As an example, take iron with $\nu = 200.$ per second. We then have

Velocity along surface,	$\frac{b}{\beta} =$		$\sqrt{\frac{\nu\mu_1}{2C_1}}$	$= 31600.$ cm. per sec.
Wave length along surface,	$\frac{2\pi}{\beta} =$		$\sqrt{\frac{2\mu_1}{\nu C_1}}$	$= 316.$ cm.
Perpendicular velocity in air,	$\frac{b}{\delta} =$	$\frac{b}{\beta} =$	$\sqrt{\frac{\nu\mu_1}{2C_1}}$	$= 31600.$ cm. per sec.
" wave length in air,	$\frac{2\pi}{\delta} =$	$\frac{2\pi}{\beta} =$	$\sqrt{\frac{2\mu_1}{\nu C_1}}$	$= 316.$ cm.
" velocity in iron,	$\frac{b}{\delta_1} =$	$\frac{b}{\mu_1\beta} =$	$\sqrt{\frac{\nu}{2C_1\mu_1}}$	$= 32.$ cm. per sec.
" wave length in iron,	$\frac{2\pi}{\delta_1} =$	$\frac{2\pi}{\mu_1\beta} =$	$\sqrt{\frac{2}{\nu C_1\mu_1}}$	$= .32$ cm.

One is impressed with the small values of all these quantities, especially the velocity of penetration into the iron, which is only 32. cm. per second. At a depth of one-fourth λ the disturbance is only one-fifth that at the surface, and in this case it is only .08 cm. or $\frac{1}{32}$ inch! Even at the depth of $\frac{1}{60}$ inch the disturbance is only about half that at the surface. The disturbance along the surface is also only half as great when the distance from the disturbing cause is 40 cm. or about 8 inches.

This theory explains an experiment of Prof. Trowbridge, in which a bar was placed in a solenoid through which an alternating current was passed. Only the outside was found to be magnetized and to attract iron filings.

From this theory we see that any iron requiring very rapid changes of magnetization must be divided up into very fine wires or laminae, in the well-known manner, whose thickness or diameter can be reduced with advantage even below $\frac{1}{100}$ inch.

The lines of magnetic force in the air are at an angle of 180°–45° with the surface, while those in the iron are nearly parallel to the surface, the tangent of the angle with the surface being $\frac{1}{\mu_1}$.

Case 2.—*Electrical Waves.*

This case, in which the electrical properties are the principal factors and the electric currents and displacement are in the plane containing the normal to the surface and the direction of propagation, is obtained by making $B = 0$ in the equations. This gives the surface conditions

$$\frac{A}{\mu}(a^2 + e^2) = \frac{A_1}{\mu_1}(a^2 + e_1^2),$$
$$Ae = A_1 e_1$$

and the conditions of wave propagation

$$a^2 + e^2 = \mu c\,(4\pi C + \kappa c), \quad a^2 + e_1^2 = \mu_1 c\,(4\pi C_1 + \kappa_1 c).$$

These give

$$a^2 = c\,(4\pi C + \varkappa c)(4\pi C_1 + \varkappa_1 c)\,\frac{\mu_1\,(4\pi C + \varkappa c) - \mu\,(4\pi C_1 + \varkappa_1 c)}{(4\pi C + \varkappa c)^2 - (4\pi C_1 + \varkappa_1 c)^2}\,,$$

$$e^2 = c\,(4\pi C + \varkappa c)^2\,\frac{\mu\,(4\pi C + \varkappa c) - \mu_1\,(4\pi C_1 + \varkappa_1 c)}{(4\pi C + \varkappa c)^2 - (4\pi C_1 + \varkappa_1 c)^2}\,,$$

$$e_1^2 = c\,(4\pi C_1 + \varkappa_1 c)^2\,\frac{\mu\,(4\pi C + \varkappa c) - \mu_1\,(4\pi C_1 + \varkappa_1 c)}{(4\pi C + \varkappa c)^2 - (4\pi C_1 + \varkappa_1 c)^2}\,.$$

The separation of the real and imaginary parts causes too complicated results, and therefore it is better to simplify them before reduction by application to the case of air and a metal. For this, $C = 0$, $\mu = 1$, $\varkappa_1 = 0$ or is at least small compared with C_1. This gives

$$\alpha = -\frac{b^2\varkappa^{\frac{3}{2}}\mu_1}{8\pi C_1} = 0 \text{ nearly,}$$

$$\beta = b\sqrt{\varkappa}\left\{1 + \frac{1}{8}\,\frac{\varkappa^2 b^2 \mu_1^2}{(4\pi C_1)^2}\right\} = b\sqrt{\varkappa} \text{ nearly,}$$

$$\gamma = \delta = -\varkappa\sqrt{\frac{b^3\mu_1}{4\pi C_1}}\,,$$

$$\gamma_1 = \delta_1 = \sqrt{2\pi C_1 b\mu_1}\,.$$

The waves thus proceed with very little damping, and have the velocity of light $\sqrt{\frac{1}{\varkappa}}$, very nearly. In each of the media, as we pass away perpendicular to the surfaces, the waves are of the diffusion type. Putting

$$\phi = \beta x + \delta y - bt,$$
$$\phi_1 = \beta x + \delta_1 y - bt,$$

the disturbance is

Vector Potential.

In the Air.

$$F = A\gamma\varepsilon^{\alpha x + \gamma y}\,(\cos\phi - \sin\phi),$$

$$G = -A\varepsilon^{\alpha x + \gamma y}\,(\alpha\cos\phi - \beta\sin\phi),$$

$$H = 0,$$

In the Metal.

$$F_1 = A\gamma\varepsilon^{\alpha x + \gamma_1 y}\,(\cos\phi_1 - \sin\phi_1),$$

$$G_1 = -A\,\frac{\varkappa b}{4\pi C_1}\,\varepsilon^{\alpha x + \gamma_1 y}(\alpha\sin\phi_1 + \beta\cos\phi_1),$$

$$H_1 = 0.$$

Magnetic Induction.

In the Air.	In the Metal.
$\bar{a} = \bar{b} = 0,$	$\bar{a}_1 = \bar{b}_1 = 0,$
$\bar{c} = Ab^2\varkappa\varepsilon^{ax+\gamma y}\cos\phi.$	$\bar{c}_1 = Ab^2\varkappa\mu_1\varepsilon^{ax+\gamma_1 y}\cos\phi_1.$

Electric Current.

$u = A\dfrac{b^2\varkappa\gamma}{4\pi}\varepsilon^{ax+\gamma y}(\cos\phi - \sin\phi),$	$u_1 = A\dfrac{C_1 b\gamma}{\sqrt{2}}\varepsilon^{ax+\gamma_1 y}(\cos\phi_1 - \sin\phi_1),$
$v = -A\dfrac{b^2\varkappa}{4\pi}\varepsilon^{ax+\gamma y}(\alpha\cos\phi - \beta\sin\phi),$	$v_1 = -A\dfrac{b^2\varkappa}{4\pi}\varepsilon^{ax+\gamma_1 y}(\alpha\cos\phi_1 - \beta\sin\phi_1),$
$w = 0.$	$w_1 = 0.$

Surface density of electricity is

$$\sigma = -A\frac{b\varkappa}{4\pi}\varepsilon^{ax}\{\beta\cos(\beta x - bt) + \alpha\sin(\beta x - bt)\}.$$

As the electricity is not at rest, there is no such thing as a potential.

The case we have solved is that of waves of electromagnetic disturbance advancing along the surface with a velocity a little less than that of light, with only a very small damping factor, accompanied with an advancing electrostatic charge upon the surface. The waves die out as one goes away from the surface, according to the very small factor $\varepsilon^{-\frac{b^2\kappa^{\frac{3}{2}}\mu_1}{8\pi C_1}y}$ in the air and to the large factor $\varepsilon^{-\sqrt{2\pi C_1 b\mu_1}y}$ in the metal. Indeed in the latter the waves are of the diffusion type which die away to $\frac{1}{535}$th part in a complete wave as discussed in the case of magnetic waves. The wave length perpendicular to the surface is $\frac{\sqrt{2}}{\sqrt{C_1\nu_1\mu_1}}$, the same as for the magnetic waves. For $\nu_1 = 200$ reversals per second, this is .32 cm. for iron and 4 cm. for copper. In other words, the current is diffused downward into the copper more than twelve times as fast as into iron. In copper, at a depth of 5 mm., the current is .45 at the surface, with 200 reversals per second; in iron the distance is only 0.4 mm. or about $\frac{1}{60}$ inch. These results are for a plane only, and in case of wires the current is diffused inward from all directions, and so much more than one-half the current reaches the center of a wire 10 mm. in diameter. I shall treat this case later.

The waves in both the air and the metal are plane waves whose intensity diminishes as one passes away from the surface. In the air, the wave surface is nearly perpendicular to the metal surface, while in the metal it is almost parallel to it. The equation to these wave surfaces is $\beta x + \delta y - bt =$ constant and $\beta x + \delta_1 y - bt =$ constant.

Two Metal Strips, Electrical Waves.

Let us now consider the case of two metal strips of infinite extent connected along one edge to the poles of an alternating dynamo. For a practical case, we can limit the strips to a certain width, provided their distance apart is small in proportion to their width. Let the strips be of the same material and the same thickness. There will then be a central plane of symmetry at which the condition is that the displacement currents and electromotive forces along it shall be zero and the magnetic force and induction perpendicular to it shall be zero.

Let the plane $y = 0$ be this plane, and the strip on the positive side be between $y = D''$ and $y = D'$. Then the vector potential in the three spaces can be written

$$\text{Central space,} \left[\begin{aligned} F &= Ae\{\varepsilon^{ey} - \varepsilon^{-ey}\}\,\varepsilon^{ax+ct}, \\ G &= -Aa\{\varepsilon^{ey} + \varepsilon^{-ey}\}\,\varepsilon^{ax+ct}, \end{aligned}\right.$$

$$\text{Metal strips,} \left[\begin{aligned} F' &= e'\{A'\varepsilon^{e'y} - B'\varepsilon^{-e'y}\}\,\varepsilon^{ax+ct}, \\ G' &= -a\{A'\varepsilon^{e'y} + B'\varepsilon^{-e'y}\}\,\varepsilon^{ax+ct}, \end{aligned}\right.$$

$$\text{Outside space,} \left[\begin{aligned} F'' &= e''A''\varepsilon^{e''y+ax+ct}, \\ G'' &= -aA''\varepsilon^{e''y+ax+ct} \end{aligned}\right.$$

These satisfy the conditions at the central plane and also the equation of continuity. For wave propagation we also have

$$a^2 = -e^2 + c^2\varkappa\mu = -e'^2 + 4\pi C'c\mu' = -e''^2 + c^2\varkappa\mu,$$

whence $$e''^2 = e^2.$$

The conditions at the surfaces of the metal are

$$c^2\varkappa A''\varepsilon^{e''D''} = 4\pi C'c\{A'\varepsilon^{e'D''} + B'\varepsilon^{-e'D''}\} \mid e''A''\varepsilon^{e''D''} = e'\{A'\varepsilon^{e'D''} - B'\varepsilon^{-e'D''}\},$$

$$4\pi C'c\{A'\varepsilon^{e'D'} + B'\varepsilon^{-e'D'}\} = c^2\varkappa A\{\varepsilon^{eD'} + \varepsilon^{-eD'}\} \mid e'\{A'\varepsilon^{e'D'} - B'\varepsilon^{-e'D'}\} = eA\{\varepsilon^{eD'} - \varepsilon^{-eD'}\}.$$

These, with the three above, give 7 equations for determining a, e, e', e'' and A'', A' and B' in terms of A.

Putting $E = \varkappa\sqrt{\frac{c^3\mu'}{4\pi C'}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1}$ and noting that eD' is very small, we have the equation

$$e^3 - Ee^2 + \frac{E}{D'}e - \frac{c^3\varkappa^2\mu'}{4\pi C'D'} = 0.$$

The first, and very near, approximation to the value of e is

$$e^2 = -\frac{E}{D'} = -\frac{\varkappa}{D'}\sqrt{\frac{c^3\mu'}{4\pi C'}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1},$$

in which $e' = \sqrt{4\pi C'c\mu'}$ very nearly.

This gives $a^2 = c^2\varkappa\mu\left\{1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{4\pi C'c}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1}\right\}$. When the thickness of the strips, $D''-D'$, is small, we have, since $\varkappa\mu = \frac{1}{v^2}$ and $\mu = 1$ nearly, for all insulators,

$$a^2 = (\alpha + i\beta)^2 = -\frac{b^2}{v^2} - \frac{ib}{4\pi C'D'(D''-D')v^2}.$$

Hence
$$\alpha^2 = \frac{b^2}{2v^2}\left\{\sqrt{\frac{1}{(4\pi C'D'(D''-D')b)^2}+1}-1\right\},$$

$$\beta^2 = \frac{b^2}{2v^2}\left\{\sqrt{\frac{1}{(4\pi C'D'(D''-D')b)^2}+1}+1\right\}.$$

The limiting case, when $C'D'(D''-D')$ is small, is

$$\beta = -\alpha = \frac{1}{v}\sqrt{\frac{b}{8\pi C'D'(D''-D')}}.$$

The waves are then of the diffusion type, giving a velocity of

$$\frac{b^2}{\beta} = v\sqrt{8\pi C'D'(D''-D')b},$$

which is the same as the solution of Sir William Thomson for a cable whose capacity and resistance per unit of length are $\frac{\varkappa}{4\pi D'}$ and $\frac{1}{C'(D''-D')}$ respectively.

But when the thickness of the strips increases in value, the damping factor α decreases and the velocity increases until we reach the limit for which

$$a^2 = -\frac{b^2}{v^2}\left\{1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\,(1+i)\right\}$$

which gives

$$\alpha^2 = \frac{b^2}{2v^2}\left\{\sqrt{1 + \frac{\mu'}{4\pi C'b\mu^2 D'^2} + \frac{2}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}} - 1 - \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\right\},$$

$$\beta^2 = \frac{b^2}{2v^2}\left\{\sqrt{1 + \frac{\mu'}{4\pi C'b\mu^2 D'^2} + \frac{2}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}} + 1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\right\}.$$

When the conductivity is large, the damping factor vanishes and the velocity of the wave is that of light, as it should be.